N 70 29928

NASA TECHNICAL TRANSLATION NASA TT F-12,962

DETERMINING THE CHARACTERISTICS OF NON-LINEAR ELEMENTS OF VIBRATION SYSTEMS FROM MOTION ANALYSIS

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Translation of "Opredeleniye Kharakteristik Nelineynykh Elementov Kolebatel'nykh Sistem iz Analiza Dvizheniya" Prikladnaya Mekhanika, Vol. 5, No. 10, 1969, pp. 1-7

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D. C. 20546 JUNE 1970

NASA TT F-12,962

DETERMINING THE CHARACTERISTICS OF NON-LINEAR ELEMENTS OF VIBRATION SYSTEMS FROM MOTION ANALYSIS

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ABSTRACT: The article proposes a mathematical method to be used in researching the nonlinear elements of real vibrating systems of a variety of different physical natures. The methodology is used to determine the characteristics of the nonlinear elements (the nonlinear restoring force and the nonlinear attenuation) of vibrating systems by analysis and special processing of information concerning a real system (based on the theory of the properties of quasi-linear vibrating systems). The purpose of this stage of research is to develop a methodology for studying the motion of a nonlinear vibrating system with one degree of freedom subjected to an external excitation force of known characteristics.

The determination and description of the characteristics of /1* nonlinear elements of real vibrating systems are of constant interest to a great many researchers. Indicative of this interest are the many articles in Soviet and foreign scientific literature, devoted to this important problem.

This article will describe the method used to determine the characteristics of the nonlinear elements (the nonlinear restorating force and the nonlinear attenuation) of vibrating systems by analysis and special processing of information concerning the actual motion of a real system.

Analysis of information concerning the motion of a nonlinear system is based on the theory of the properties of quasi-linear vibrating systems. Certain aspects of earlier publications [2,3, 6,7] were used in this analysis.

The purpose of this stage of the research is to develop a

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* Numbers in the right margin indicate pagination in the foreign text.

methodology for analyzing information concerning the motion of a nonlinear vibrating system with one degree of freedom, subjected to an external excitation force of known characteristics. In addition the necessary program for the vibration tests of the system studies has been established.

The following differential equation

$$mx + f_1(x) + f_2(x) = E \sin \nu t. \qquad (1)$$

can be used to describe the forced motion of a system subjected to a sinusoidal force. From [5], the functions f_i (z) (i = 1,2) can be presented in the form

$$f_i(z) = \bar{f}_i(z) + \hat{f}_i(z) \quad (i = 1, 2);$$

$$\bar{f}_i(z) = -\bar{f}_i(-z); \quad \hat{f}_i(z) = \hat{f}_i(-z). \qquad (2)$$

In addition, we will assume that f_i (0) = 0, f_i (z) $\gtrless$ 0 when z $\gtrless$ 0 (i = 1,2). Physically, the functions f_1 (z) and f_2 (z) are such that the system can complete what are approximately sinusoidal forced vibrations when the excitation force is sinuosoidal.

Thus, five magnitudes: m, $\bar{f}_i$(z), $\hat{f}_i$(z) (i = 1,2) must be determined from the experimental data, providing the exact E and ν characteristics of the sinusoidal excitation force are known.

To solve this problem, let us compute the magnitude of the average displacement of the system over the period of one vibration, and let us also plot the amplitude versus phase curves for the rate, x, at which it is moving. By the amplitude versus phase curve /2 (APC) for the system under study we understand the APC to mean the fundamental harmonic of its reaction to the sinusoidal excitation.

Some approximation method is needed to solve Equation 1 in order to study the properties of the APC for the velocity of the original nonlinear system. The amplitude versus frequency, and phase versus frequency curves should be derived from the solution obtained. Let us find the APC for the fundamental harmonic of the system's velocity, eliminating the frequency from the relationships deduced.

In accordance with the Krylov-Bogolyubov-Mitropol'skiy method [1], the fundamental harmonic of the velocity of steady-state forced

vibrations of the nonlinear system considered can be written in the form

$$x = a\nu \sin(\nu t + \theta), \tag{3}$$

where

$$a = \frac{E}{\sqrt{[c(a) - m\nu^2]^2 + b^2(v)\nu^2}}; \tag{4}$$

$$\operatorname{tg}\theta = \frac{c(a) - m\nu^2}{\nu b(v)}; \quad v = a\nu; \tag{5}$$

c(a) and b(v) are equivalent linearized rigidity and damping factors, respectively, determined through the formulas

$$c(a) = \frac{1}{\pi a}\int_0^{2\pi} \bar{f}_2(a\cos\psi)\cos\psi\, d\psi = \frac{4}{\pi a}\int_0^{\pi/2} \bar{f}_2(a\cos\psi)\cos\psi d\psi; \tag{6}$$

$$b(v) = \frac{1}{\pi v}\int_0^{2\pi} \bar{f}_1(v\sin\psi)\sin\psi d\psi = \frac{4}{\pi v}\int_0^{\pi/2} \bar{f}_1(v\sin\psi)\sin\psi d\psi. \tag{7}$$

We obtain the following expression for the APC for the velocity by eliminating the frequency, ν, from the relationships at (4) and (5):

$$v = \frac{E\cos\theta}{b(v)}. \tag{8}$$

The plot of the dependency at (8) in polar coordinates r and θ is an oval, the shape of which depends on the behavior of the magnitude b(v). In the special case when b(v) = b = constant, the relationship at (8) is the polar equation of a circle with the pole selected on the circle proper and the axis passing through its center. The diameter of this circle is equal to E/b (Figure 1a). The frequency at which the APC plot intersects the polar axis is determined from the condition that tan θ = 0, or on the basis of the first formula at (5)

$$\nu^2 = \frac{c(a)}{m}. \tag{9}$$

Thus, the APC plot will intersect the polar axis at different frequencies for different excitation levels if c (a) ≠ constant.

It should be noted that in the general case of a nonlinear system with a sufficiently high level of excitation force amplitude, not all of the APC plot can be obtained experimentally.

There are frequencies, ν, at which the amplitudes of the forced vibrations become unstable. These frequencies can be determined theoretically through the equation

$$\frac{d\nu}{da} = 0,$$

where ν is an implicit function of a, found through the equality /3

$$a^2\{[c(a) - m\nu^2]^2 + b^2(v)\nu^2\} - E^2 = 0.$$

Let us use curves of the APC for the velocity obtained at two different excitation levels (Figure 1b), to find the discrete parameter m. When $V = V_o$, we have for the curve corresponding to the excitation level $E = E_1$.

$$\operatorname{tg}\theta(\nu_0) = 0; \quad v_0 = \frac{E_1}{b(v_0)} = OD = l.$$

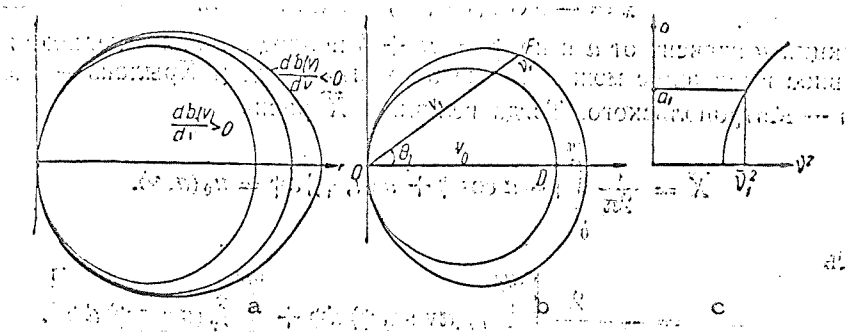

Figure 1.

In the case of $E = E_2$, let us lay off the segment OF from the origin of the polar coordinates, the length of which is $OF = l = v_1 = a_1\nu_1$. Let us designate the angle formed by this segment and the polar axis as θ_1. Then, from the relationship at (9), let us find $\bar{\nu}_1^2$ in terms of a_1 (Figure 1c). The magnitudes θ_1, $\bar{\nu}_1^2$, ν_1^2 found in this way satisfy the relationship

$$m = \frac{\operatorname{tg}\theta_1\nu_1 E_1}{(\nu_1^2 - \bar{\nu}_1^2)l}. \qquad (10)$$

Let us introduce a change of variables $a\cos\psi = v\sin\psi = z$

in the equalities at (6) and (7) to find the functions $\overline{f}_1(\dot{x})$ and $\overline{f}_2(x)$. We obtain the result

$$c(a) = \frac{4}{\pi a^2}\int_0^a \frac{\overline{f}_2(z)\,z}{\sqrt{a^2-z^2}}\,dz; \qquad b(v) = \frac{4}{\pi v^2}\int_0^v \frac{\overline{f}_1(z)\,z}{\sqrt{v^2-z^2}}\,dz. \tag{11}$$

Thus, if c(a) and b(v) are known as functions of the amplitudes of the displacement and the velocity, by solving the integral equations at (9) we can find $\overline{f}_1(x)$ and $\overline{f}_2(x)$.

The Abel integral equation with respect to the first derivatives of the sought for functions are obtained by integration by parts of any of the relationship at (11), with the equations at (2) taken into consideration. Since the solution to the latter is well known [4], we find finally for $f_1(\dot{x})$ and $\overline{f}_2(x)$

$$\overline{f}_1(\dot{x}) = \int_0^{\dot{x}} \frac{d}{d(v^2)}[v^2 b(v)]\frac{v\,dv}{\sqrt{\dot{x}^2-v^2}}; \qquad \overline{f}_2(x) = \int_0^x \frac{d}{d(a^2)}[a^2 c(a)]\frac{a\,da}{\sqrt{x^2-a^2}}. \tag{12}$$

Once the m parameter is determined experimentally through the formula at (10), we obtain the curves for the dependencies c (a) and b (v) through the relationship at (8) and (9) by testing the system in the phase resonance made at different excitation levels, E. Since we have the equalities at (12), the problem of making the experimental determination of the odd components of the sought for functions $f_1(\dot{x})$ and $f_2(x)$ is solved. /4

If spatial lack of symmetry of vibrations is observed during the experiments (which is easily detectable, taking the rate of displacement $\dot{x}$, and the magnitude (x) for system reaction), the even components of the functions $f_1(\dot{x})$ and $f_2(x)$ must be determined.

To solve this problem let us compute the magnitude $\overline{X}$ for the mean displacement of the vibrating mass during one vibration period. The solution to Equation (1), with all harmonics taken into consideration is, in the form

$$x = -a\cos(\nu t + \theta) + u(a, \nu t + \theta).$$

The function u depends on a, and when $\psi = \nu t + \theta$ is periodic with

period 2π. Its explicit expression can be obtained by the Krylov-Kogolyubov-Mitropol'skiy method as well. Then the magnitude $\overline{X}$

$$\overline{X} = \frac{1}{2\pi}\int_0^{2\pi}[-a\cos\psi + u(a,\psi)]\,d\psi = u_0(a,\nu). \tag{13}$$

Here

$$u_0(a,\nu) = -\frac{2}{c(a)\pi}\left[\int_0^{\pi/2}\hat{f}_1(a\nu\sin\psi)\,d\psi + \int_0^{\pi/2}\hat{f}_2(a\cos\psi)\,d\psi\right].$$

Any of the two integrals in the right side of the equality at (12) can be reduced to the following form by substituting $z = a\nu\sin\psi$ ($z = a\cos\psi$)

$$\int_0^{\pi/2}\hat{f}_1(a\nu\sin\psi)\,d\psi = \int_0^{a\nu}\hat{f}_1(z)\,w(a\nu,z)\,dz; \quad \int_0^{\pi/2}\hat{f}_2(a\cos\psi)\,d\psi = \int_0^{a}\hat{f}_2(z)\,w(a,z)\,dz, \tag{14}$$

where

$$w(a,z) = (a^2 - z^2)^{-\frac{1}{2}}. \tag{15}$$

Considering the equality at (14), we can write the relationship at (13) as

$$\int_0^{a\nu}\hat{f}_1(z)\,w(a\nu,z)\,dz + \int_0^{a}\hat{f}_2(z)\,w(a,z)\,dz = -\frac{\pi}{2}c(a)\,u_0(a,\nu). \tag{16}$$

The right side of the equality at (16) is a known function of the argument of a, on an experimental basis, and depends on ν, as well as on the parameter. The functions $\hat{f}_1(x)$ and $\hat{f}_2(x)$ can be determined through the use of the relationship at (16). Let us test the system under study in the harmonic vibration mode at two fixed frequencies ν_j ($j=1,2$) and various excitation equations $0<E\leqslant E_{max}$ range. The result will be a system of two Abel integral equations for use in determining the remaining functions $\hat{f}_1(z)$ and $\hat{f}_2(z)$

$$\int_0^{a\nu_j}\hat{f}_1(z)\,w(a\nu_j,z)\,dz + \int_0^{a}\hat{f}_2(z)\,w(a,z)\,dz = -\frac{\pi}{2}c(a)\,u_0(a,\nu_j) \quad (j=1,2). \tag{17}$$

The functions $\hat{f}_1(z)$ and $\hat{f}_2(z)$ must be sought for in the form of polynomials of degrees a and $a\nu_j$, if the terms in the right side of the equality at (17) are expressed as polynomials of degree a. Substituting these polynomials in the equality at (17), and equating the coefficients with identical degrees, a, we obtain a system of /5 algebraic equations for determining the coefficients of the sought-for expansions. Thus, if $-\frac{\pi}{2} c(a) u_0(a, \nu_j) = \Sigma Q_{kj} a^k (j = 1, 2)$, then $\hat{f}_1(\dot{x}) = \Sigma M_k |\dot{x}|^k$; $f_2(x) = \Sigma N_k |x|^k$.

The coefficients M_k and N_k can be found through the system of algebraic equations.

$$Q_{kj} = \xi_k (\nu_j^k M_k + N_k) \quad (k = 1, 2 \ldots n; \quad j = 1, 2),$$

where ξ_k are the known coefficients, whose values for k = 1,2...5 are $\xi_1 = 1$, $\xi_2 = \pi/4$, $\xi_3 = 2/3$, $\xi_4 = 3\pi/16$ and $\xi_5 = 8/15$.

Let us take some examples to illustrate the method described.

Example 1. Find the parameters of a vibrating system which produced the following vibration test data

$$c(a) = k + \alpha a^2; \quad b(v) = b = \text{const}; \quad u_0(a, \nu) c(a) = a^2 (\nu^2 \delta + \gamma).$$

From the second equation at (12), we have

$$\bar{f}_2(x) = \int_0^x \frac{d}{d(x^2)} (ka^2 + \alpha a^4) \frac{ada}{\sqrt{x^2 - a^2}} = kx + \frac{4}{3} \alpha x^3.$$

In precisely the same manner, it can be shown that $\bar{f}_1(x) = bx$.

The equation at (17) can be used to find the $\hat{f}_1(x)$ and $\hat{f}_2(x)$

$$\int_0^{a\nu} \hat{f}_1(z) w(a\nu, z) + \int_0^a \hat{f}_2(z) w(a, z) dz = -\frac{\pi}{2} a^2 (\delta \nu^2 + \gamma).$$

This equality is possible only if

$$\int_0^{a\nu} f_1(z) w(a\nu, z) dz = -\frac{\pi}{2} \delta a^2 \nu^2; \tag{18}$$

$$\int_0^a f_2(z) w(a, z) dz = -\frac{\pi}{2} \gamma a^2. \tag{19}$$

We can, after taking the notation at (15) into consideration, write the solution for the integral equation at (18) in the form

$$\hat{f}_1(v) = -\delta \frac{d}{dv} \int_0^v \frac{z^3}{\sqrt{v^2 - z^2}}\, dz = -2\delta v^2.$$

Consequently,

$$f_1(\dot{x}) = -2\delta \dot{x}^2.$$

We find similarly

$$\hat{f}_2(x) = -2\gamma x^2.$$

Thus, the vibrating system can be described by the nonlinear equation /6

$$m\ddot{x} + b\dot{x} - 2\delta \dot{x}^2 + kx - 2\gamma x^2 + \frac{4}{3}\alpha x^3 = E \sin \nu t.$$

Example 2. Let us find the parameters of a vibrating system from the following data, obtained during vibration tests

c (a) = k = constant; b (v) = b = constant

The expression $u_o(a,\nu)c(a) = \varepsilon a$ is independent of the frequency. In that case it can be shown that $f_1(x) = bx$, and $f_2(x) = kx$.

Since the magnitude of the systematic force acting on the vibrating system is independent of the frequency, ν, the equality at (16) for this case can be written as

$$\int_0^a \hat{f}_2(z) \frac{dz}{\sqrt{a^2 - z^2}} = -\frac{\pi}{2} \varepsilon a, \qquad (20)$$

The solution to the integral equation at (20) is in the form

$$\hat{f}_2(a) = -\frac{d}{da} \varepsilon \int_0^a \frac{z^2}{\sqrt{a^2 - z^2}}\, dz = -\varepsilon \frac{\pi}{2} a,$$

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$$\hat{f}_2(x) = -\varepsilon \frac{\pi}{2} |x|.$$

The vibrating system can be described by the following equation:

$$m\ddot{x} + b\dot{x} + kx - \varepsilon \frac{\pi}{2}|x| = E \sin \nu t.$$

The graph of the restoring force, $f_2(x)$, is shown in Figure 2a. The tangents of the angles of inclination of the characteristic curve are associated by the relationships

$$\operatorname{tg} \alpha_1 = k - \varepsilon \frac{\pi}{2}; \quad \operatorname{tg} \alpha_2 = k + \varepsilon \frac{\pi}{2}.$$

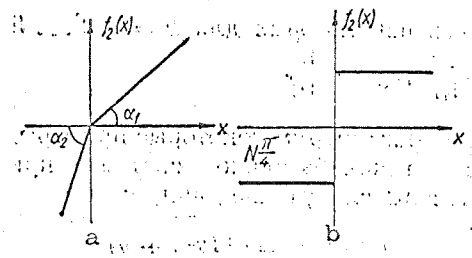

Figure 2.

Example 3. Let us determine the dynamic characteristics of the system using the experimental vibration test data

$$c(a) = \frac{N}{a}; \quad b(v) = b = \text{const}; \quad u_0(a, \nu) \cdot c(a) \equiv 0. \tag{21}$$

The first of the conditions at (21), based on the second equation at (12) for the function $\overline{f}_2(x)$, gives

$$f_2(x) = \int_0^x \frac{d}{d(a^2)} \left(a^2 \frac{N}{a} \right) \frac{ada}{\sqrt{x^2 - a^2}} = \frac{N\pi}{4};$$

While the second and third conditions result in the relationships /7

$$\overline{f}_1(\dot{x}) = b\dot{x}; \quad \hat{f}_1(\dot{x}) = \hat{f}_2(x) \equiv 0.$$

The graphic of the restoring force is shown in Figure 2b.

The examples cited are indicative of the great scope of the proposed method, which is suitable for the experimental study of vibrating systems of different physical natures.

Received
January 31, 1969

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Translated for the National Aeronautics and Space Administration under contract No. NASw-2038 by Translation Consultants, Ltd., 944 South Wakefield Street, Arlington, Virginia 22204.